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Symmetry and conservation: inverse Noether's theorem and general formalism

Joe Rosen
Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel

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Abstract. The previously developed 'continuous temporal development and Noether's theorem' approach to the association of symmetries and conservations of the laws of nature, part of the foundation of any (classical or quantum) Lagrangian theory, is generalised significantly. Within this framework a strong result is obtained for the inverse Noether's theorem, associating symmetries with given conservations. A general symmetry-conservation formalism is proposed, into which both the previously developed 'linear temporal development and conserved eigenheit' approach, part of the foundation of any quantum theory and also applicable to certain classical ones, and the 'continuous temporal ...' approach, as different as they are, fit nicely.

1. Introduction

Nature exhibits symmetries and conservations; we associate symmetry and conservation in our theories. In a previous article (Rosen and Freundlich 1978), referred to hereafter as SAC, an investigation motivated by two premises based on this fact was presented. The first premise is that the most important handle to the laws of nature that we have, as long as the laws of nature themselves are not fully known (if, indeed, they ever will be or can be), is their symmetries and conservations. The second premise is that, although it is we who are associating symmetries and conservations, and not nature, the symmetry–conservation correspondences that emerge from our physical theories and formalisms are no coincidence, but are one of the most fundamental, if not *the* most fundamental, aspects of the laws of nature, barring complete knowledge of the laws themselves.

These two premises motivate us to find and investigate theoretical structures that link symmetries and conservations, including paring down known theoretical structures to their barest essentials, in order to reveal the vital kernel relating symmetries and conservations of the laws of nature. This kernel could then serve as a nucleus around which present theories might be restructured or new theories constructed. And, more fundamentally, this kernel could help us learn something about the nature of nature.

Thus we pose the question: what are minimal sets of assumptions giving relations between symmetries and conservations? In partial answer we presented, examined and exemplified in SAC two sets of assumptions relating symmetry and conservation that we have not succeeded in reducing, so they might be minimal. One set we called 'linear temporal development and conserved eigenheit', referred to hereafter as LTDCE. It is a subset of the full set of assumptions underlying any quantum theory and is also applicable to certain classical theories. The other set of assumptions is the 'continuous temporal development and Noether's theorem' approach, referred to hereafter as CTDNT. It forms part of the set of assumptions of any (classical or quantum) Lagrangian theory. The results found in SAC for LTDCE and CTDNT are summarised and discussed there in § 8.

In the present article we advance the investigation of symmetry and conservation that was started in SAC. In § 2 we generalise CTDNT significantly by allowing more general transformation groups. In § 3 we investigate the inverse Noether's theorem, and a strong result is obtained. In § 4 we work through a detailed example to clarify §§ 2 and 3. Section 5 summarises our CTDNT results. In § 6 we propose a general symmetry–conservation formalism, into which both LTDCE and CTDNT, as different as they are, fit nicely.

In order to save space we sacrifice self-containment of the present article and rely heavily on the notation and results of SAC. As in SAC, we assume whatever technical, mathematical assumptions are necessary to make our discussion meaningful.

2. More on continuous temporal development and Noether's theorem (CTDNT)

We generalise the one-parameter transformation groups $(M_\epsilon, f_{M_\epsilon}(t))$ considered in § 4 of SAC to $(M_{\epsilon,t}, f_\epsilon(u, t))$. The ϵ-parametrised group of mappings of state space onto itself, M_ϵ, is generalised to depend additionally on the time t of the object state, $M_{\epsilon,t}$, so that state u at time t is mapped to state $M_{\epsilon,t}u$. The time of the image state $M_{\epsilon,t}u$ is now given by $f_\epsilon(u, t)$, where the corresponding function in SAC, $f_{M_\epsilon}(t)$, is generalised to include additionally dependence on the object state itself. The function $f_\epsilon(N(t, t_1)u_1, t_1)$ must be strictly monotonically ascending in t for all u_1, t_1, ϵ, where $N(t, t_1)$ is the temporal development mapping defined in § 4 of SAC, and suitable generalisations of equations (28)–(33) of SAC must hold.

The development of CTDNT presented in §§ 4 and 6 of SAC, including figures 2, 4 and 5 there, generalises in a very straightforward manner, when transformation groups $(M_{\epsilon,t}, f_\epsilon(u, t))$ are used instead of $(M_\epsilon, f_{M_\epsilon}(t))$. So we shall refrain from going through the development again. However, due to its importance we give the generalised version of the commutativity relation expressing the fact that $(M_{\epsilon,t}, f_\epsilon(u, t))$ is a symmetry of the laws of nature in the sense of SAC:

$$M_{\epsilon,t}N(t, t_1)u_1 = N(f_\epsilon(N(t, t_1)u_1, t), f_\epsilon(u_1, t_1))M_{\epsilon,t_1}u_1 \tag{1}$$

for all u_1, t_1, t, ϵ.

In the type of investigation with which we are involved it is always very beneficial to generalise; the more, the merrier. The above generalisation of LTDNT, however, is not only worthwhile for its own sake, but is also useful, having implications for the inverse Noether's theorem and giving the following result, which also has further implications.

Let $(M_{\epsilon,t}, f_\epsilon(u, t))$ be a group of transformations such that the generalisation of equation (41) of SAC holds,

$$[\delta_{(M,f)}L] = \epsilon\frac{\mathrm{d}}{\mathrm{d}t}S_{(M,f)}(N(t, t_1)u_1, t), \tag{2}$$

where ϵ is infinitesimal,

$$[\delta_{(M,f)}L] \overset{\mathrm{def}}{=} \delta_{(M,f)}L + \epsilon\frac{\mathrm{d}}{\mathrm{d}t}[f'(N(t, t_1)u_1, t)L(N(t, t_1)u_1, t)] \tag{3}$$

(generalisation of equation (39) of SAC), $L(u, t)$ being the Lagrangian,

$$f'(u, t) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\epsilon} f_\epsilon(u, t)|_{\epsilon=0} \tag{4}$$

(generalisation of equation (40) of SAC), and the term $\delta_{(M,f)}L$ in equation (3) is defined to be $\delta_{N_{(M,f)}}L$, where the mapping $N_{(M,f),\epsilon}(t, t_1)$ is the variation of $N(t, t_1)$ associated with $(M_{\epsilon,t}, f_\epsilon(u, t))$ through

$$N_{(M,f),\epsilon}(f_\epsilon(N(t, t_1)u_1, t), t_1)u_1 \stackrel{\text{def}}{=} M_{\epsilon,t}N(t, t_1)u_1 \tag{5}$$

(generalisation of equation (35) of SAC), and $\delta_{N_{(M,f)}}L$ is the variation of L produced by $N_{(M,f),\epsilon}(t, t_1)$,

$$\delta_{N_{(M,f)}}L \stackrel{\text{def}}{=} \epsilon\left(\frac{\mathrm{d}}{\mathrm{d}t}\Pi_{N_{(M,f)}}(N(t, t_1)u_1, t) + L'_{N_{(M,f)}}\left(N(t, t_1)u_1, t, \frac{\mathrm{d}}{\mathrm{d}t}\right)\right) \tag{6}$$

(generalisation of equation (22) of SAC). Thus $(M_{\epsilon,t}, f_\epsilon(u, t))$ is associated with the conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{(M,f)}(N(t, t_1)u_1, t) \stackrel{\circ}{=} 0, \tag{7}$$

where

$$Q_{(M,f)}(u, t) = S_{(M,f)}(u, t) - f'(u, t)L(u, t) - \Pi_{(M,f)}(u, t) \tag{8}$$

(generalisation of equation (44) of SAC) up to an arbitrary additive constant, and $\Pi_{(M,f)}$ is defined to be $\Pi_{N_{(M,f)}}$ in equation (6). (The duplication of symbols, with $\delta_{(M,f)} \stackrel{\text{def}}{=} \delta_{N_{(M,f)}}$ and $\Pi_{(M,f)} \stackrel{\text{def}}{=} \Pi_{N_{(M,f)}}$, is a result of the way CTDNT was developed in § 4 of SAC.)

It might happen that for certain symmetry groups for which equation (2) holds the equation takes the form

$$[\delta_{(M,f)}L] = 0, \tag{9}$$

so that $S_{(M,f)}(u, t)$ for these symmetry groups is constant and can be taken as equal to 0. (In Rosen (1972) such transformations in the context of classical field theory were called 'invariance transformations'.) In § 7 of SAC we saw an advantage for such symmetry transformations (equation (107) and the following there), and another advantage appears later in the present article. We know that while $\Pi_{(M,f)}(u, t)$, $Q_{(M,f)}(u, t)$ and $R_{(M,f)}(u, t)$, where

$$R_{(M,f)}(u, t) = S_{(M,f)}(u, t) - f'(u, t)L(u, t), \tag{10}$$

are the same functions of u, t for all members of the Noether family (i.e. all equivalent transformation groups in the sense of § 6 of SAC) of symmetry group $(M_{\epsilon,t}, f_\epsilon(u, t))$ obeying equation (2), $S_{(M,f)}(u, t)$ is not. If $(M_{\epsilon,t}, f_\epsilon(u, t))$ obeys equation (2) but not equation (9), can another member of its Noether family, all of which are also symmetry groups and obey equation (2), be found for which equation (9) holds? The answer is that within the framework of CTDNT, as developed in SAC and generalised above, the possibility exists but is not assured, as we now show.

Let $(\bar{M}_{\epsilon,t}, \bar{f}_\epsilon(u, t))$ be a member of the Noether family of $(M_{\epsilon,t}, f_\epsilon(u, t))$. Then

$$\bar{M}_{\epsilon,t}u = N(\bar{f}_\epsilon(u, t), f_\epsilon(u, t))M_{\epsilon,t}u \tag{11}$$

(generalisation of equation (58) of SAC). Clearly for $(\bar{M}_{\epsilon,t}, \bar{f}_\epsilon(u,t))$ to obey equation (9), given that $(M_{\epsilon,t}, f_\epsilon(u,t))$ obeys equation (2), it is necessary and sufficient that

$$\bar{f}'(u,t) = f'(u,t) - S_{(M,f)}(u,t)/L(u,t). \tag{12}$$

Assuming that $L(u,t)$ does not vanish, we proceed to solve for $\bar{f}_\epsilon(u,t)$, given $f_\epsilon(u,t)$, $S_{(M,f)}(u,t)$ and $L(u,t)$, and find solutions, such as $f_\epsilon - \epsilon S_{(M,f)}/L$, $f_\epsilon \pm [1-\exp(\pm\epsilon S_{(M,f)}/L)]$, $f_\epsilon \pm [1-\exp(\pm\epsilon S_{(M,f)})]/L$, $f_\epsilon = \exp(-\epsilon S_{(M,f)}/tL)$. However, we have no way of guaranteeing in general that $\bar{f}_\epsilon(N(t,t_1)u_1,t)$ be strictly monotonically ascending in t. Thus we have the unassured possibility that $(\bar{M}_{\epsilon,t}, \bar{f}_\epsilon(u,t))$ obeying equation (9) can be found.

As an example, we take classical Lagrangian mechanics, following § 7 of SAC (with appropriate changes in notation). Let $(M_{\epsilon,t}, f_\epsilon(u_i, \dot{u}_j, t))$ transform the trajectory $u_i(t)$ to trajectory $u_i(t) + \epsilon v_{(M,f)i}$, where state $(u_i(t), \dot{u}_j(t))$ on the object trajectory at time t is mapped to state $(u_i(t) + \epsilon w_{(M,f)i}(u_k(t),t),\ \dot{u}_j(t) + \epsilon \dot{w}_{(M,f)j}(u_k(t),t))$ on the image trajectory at time

$$f_\epsilon(u_i(t), \dot{u}_j(t), t) = t + \epsilon f'(u_i(t), \dot{u}_j(t), t), \tag{13}$$

and

$$v_{(M,f)i} = w_{(M,f)i}(u_j(t),t) - f'(u_j(t), \dot{u}_k(t), t)\dot{u}_i(t). \tag{14}$$

Let $(M_{\epsilon,t}, f_\epsilon(u_i, \dot{u}_j, t))$ obey equation (2) (in the form of equation (97) of SAC). Then for $(\bar{M}_{\epsilon,t}, \bar{f}_\epsilon(u_i, \dot{u}_j, t))$ to be a member of the same Noether family and to obey equation (9) it is necessary that

$$\bar{f}'(u_i, \dot{u}_j, t) = f'(u_i, \dot{u}_j, t) - S_{(M,f)}(u_i, \dot{u}_j, t)/L(u_i, \dot{u}_j, t). \tag{15}$$

3. Inverse Noether's theorem

First of all we note the following. We have seen that a transformation group $(M_{\epsilon,t}, f_\epsilon(u,t))$ defines a unique variation of $N(t,t_1)$, $N_{(M,f),\epsilon}(t,t_1)$, by equation (5), and we have seen in § 6 of SAC that equivalent transformation groups define the same variation of $N(t,t_1)$. Thus it is the whole family of equivalent transformation groups, called a Noether family if it is associated with a conservation, that is associated with the variation of $N(t,t_1)$. Conversely, the generalisation of CTDNT presented in the preceding section allows any variation of $N(t,t_1)$, $N_\epsilon(t,t_1)$, to define a unique family of equivalent transformation groups, whose time-non-varying member $(\bar{M}_{\epsilon,t}, t)$ is given by

$$\bar{M}_{\epsilon,t} = N_\epsilon(t,t), \tag{16}$$

and whose general member $(M_{\epsilon,t}, f_\epsilon(u,t))$ with arbitrary $f_\epsilon(u,t)$ is given by

$$M_{\epsilon,t}u = N(f_\epsilon(u,t),t)N_\epsilon(t,t)u \tag{17}$$

according to equation (11).

Now we are prepared to attack the inverse Noether's theorem, obtaining a symmetry from a given conservation. Within the framework of CTDNT the following can be stated. If we have a function $Q(u,t)$ that is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(N(t,t_1)u_1,t) \overset{\mathrm{o}}{=} 0 \tag{18}$$

for all u_1, t_1, and if a variation of $N(t, t_1)$, $N_\epsilon(t, t_1)$, can be found for which

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(N(t, t_1)u_1, t) = L'_{N_\epsilon}\left(N(t, t_1)u_1, t, \frac{\mathrm{d}}{\mathrm{d}t}\right) \tag{19}$$

then the members of the Noether family of transformation groups defined by $N_\epsilon(t, t_1)$ are all symmetry groups.

To see that, note that from equation (22) of SAC and equation (19) above follows the validity of equation (24) of SAC and of the generalisation of equation (63) of SAC for the time-non-varying member of the Noether family defined by $N_\epsilon(t, t_1)$. Then reverse the line of reasoning from equation (59) to equation (63) in § 6 of SAC (analogous to the line of reasoning from equation (34) to equation (41) in § 4 of SAC, the latter presented in more detail) to obtain the result that the transformation group

$$(\bar{M}_{\epsilon,t}, t) = (N_\epsilon(t, t), t) \tag{20}$$

is a symmetry and thus are all members of its Noether family. (Since we now know that every Noether family is a symmetry family, the relevant statement in the sixth paragraph of § 6 of SAC should be tightened up accordingly, to make it reflect the situation of the generalised CTDNT framework.)

That is what can be definitely stated about the inverse Noether's theorem. We now consider possibilities. When will equation (19) hold for the conserved function $Q(u, t)$? Given equation (18), if equation (19) does not hold, then there must exist a variation of $N(t, t_1)$, $N_\epsilon(t, t_1)$, and a real function of a real variable, $F(\ )$ such that $F(0) = 0$, for which

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(N(t, t_1)u_1, t) = F\left(L'_{N_\epsilon}\left(N(t, t_1)u_1, t, \frac{\mathrm{d}}{\mathrm{d}t}\right)\right). \tag{21}$$

If $F(\ )$ is invertible, we have

$$F^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} Q(\ ,\ )\right) = L'_{N_\epsilon}(\ ,\ ,\ ). \tag{22}$$

If $F(\ )$ is such that

$$F^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} Q(\ ,\ )\right) = \frac{\mathrm{d}}{\mathrm{d}t} \bar{Q}(\ ,\ ) \tag{23}$$

for some $\bar{Q}(u, t)$, we obtain equation (19) with $\bar{Q}(u, t)$ substituted for $Q(u, t)$, and the inverse Noether's theorem applies.

As an example of the inverse Noether's theorem, we again take classical Lagrangian mechanics, following § 7 of SAC (with the notation modified appropriately). Let $Q(u_i, \dot{u}_j, t)$ be conserved (equation (18) holds in the form of equation (83) of SAC), such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(u_j(t), \dot{u}_k(t), t) = a_i(u_j(t), \dot{u}_k(t), t) E^i L. \tag{24}$$

Then putting

$$v_{N_\epsilon i} = a_i, \tag{25}$$

we define a variation $N_\epsilon(t, t_1)$ for which equation (82) of SAC holds. It then follows that equation (2) (in the form of equation (97) of SAC) holds for the time-non-varying transformation group $(N_\epsilon(t, t), t)$ and that the latter is a symmetry and thus so are all members of its Noether family. To obtain a member for which equation (90) of SAC holds, find an appropriate $f_\epsilon(u_i, \dot{u}_j, t)$ obeying

$$\partial a_i/\partial \dot{u}_j + u_i\,\partial f'/\partial \dot{u}_j + f'\,\delta_i^j = 0 \tag{26}$$

by equation (14) (equation (89) of SAC). To obtain a member for which equation (9) holds, find an appropriate $f_\epsilon(u_i, \dot{u}_j, t)$ obeying

$$f' = -(Q + a_i\pi^i)/L \tag{27}$$

by equation (98) of SAC and equation (15) above.

4. Example

To help clarify the previous two sections we shall now work through a rather specific example step by step. The example chosen is hopefully both sufficiently simple, so that it indeed clarifies things as intended, and complicated enough to do this non-trivially. Consider a set of interacting point particles in one dimension, where indices i, j, etc, label the particles, and the summation convention is not in effect. Conditions are non-relativistic. The interaction potential is assumed to depend only on the distances between the particles and the differences of their velocities. Thus the Lagrangian is

$$L = \tfrac{1}{2}\sum_i m_i\dot{x}_i^2 - V(x_i - x_j, \dot{x}_k - \dot{x}_l), \tag{28}$$

where m_i is the (constant) mass of the ith particle, x_i is its spatial coordinate, and $\dot{x}_i$ its velocity. A state of the system at any time t is completely described by the coordinates and velocities of all the particles at that time (see § 7 of SAC). The equations of motion are (from equations (78)–(81) of SAC)

$$m_i\ddot{x}_i \overset{\mathrm{O}}{=} F_i, \tag{29}$$

for all i, where $\ddot{x}_i$ is the acceleration of the ith particle, upon which the force F_i acts, where

$$F_i \overset{\mathrm{def}}{=} \sum_{j\neq i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial(\dot{x}_i - \dot{x}_j)} - \frac{\partial V}{\partial(x_i - x_j)}\right). \tag{30}$$

We cannot, of course, give an explicit representation of the temporal development mapping $N(t, t_1)$ for finite time intervals for this general potential. However, we can do so for infinitesimal time intervals. Thus the state $(x_i(t), \dot{x}_j(t))$ at time t develops into the state $(x_i(t+\mathrm{d}t), \dot{x}_j(t+\mathrm{d}t))$ after time interval $\mathrm{d}t$:

$$\begin{aligned}(x_i(t), \dot{x}_j(t)) &\xrightarrow{N(t+\mathrm{d}t,t)} (x_i(t+\mathrm{d}t), \dot{x}_j(t+\mathrm{d}t)) = (x_i(t) + \dot{x}_i(t)\,\mathrm{d}t, \dot{x}_j(t) + \ddot{x}_j(t)\,\mathrm{d}t)\\ &= (x_i(t) + \dot{x}_i(t)\,\mathrm{d}t, \dot{x}_j(t) + (F_j/m_j)\,\mathrm{d}t).\end{aligned} \tag{31}$$

In addition, consider the one-parameter transformation group $(M_{\epsilon,t}, f_\epsilon)$ of non-relativistic boosts:

$$(x_i(t), \dot{x}_j(t)) \xrightarrow{M_{\epsilon,t}} (x_i(t) + \epsilon t, \dot{x}_j(t) + \epsilon) \tag{32}$$

$$f_\epsilon = t \tag{33}$$

defined for all boost velocities ϵ.

Due to the assumed structure of V, equation (1) is found to hold, where its left-hand side for this example is $M_{\epsilon,t+\mathrm{d}t}N(t+\mathrm{d}t, t)(x_i(t), \dot{x}_j(t))$, its right-hand side is $N(t+\mathrm{d}t, t)M_{\epsilon,t}(x_i(t), \dot{x}_j(t))$, and the two are calculated using equations (30)–(32). Thus the group of boosts is a symmetry for the example.

Let us check if equation (2) holds. First we see what the mapping $N_{(M,t+\mathrm{d}t),\epsilon}(t+\mathrm{d}t, t)$, the variation of $N(t+\mathrm{d}t, t)$ associated with the boost group, looks like. By equation (5), and using equations (31) and (32), it is

$$(x_i(t), \dot{x}_j(t)) \xrightarrow{N_{(M,t+\mathrm{d}t),\epsilon}(t+\mathrm{d}t,t)} (x_i(t)+\epsilon t+(\dot{x}_i(t)+\epsilon)\,\mathrm{d}t, \dot{x}_j(t)+\epsilon+(F_j/m_j)\,\mathrm{d}t). \tag{34}$$

Now equation (4) gives us

$$f'=0. \tag{35}$$

For δL we obtain, using equation (73) of SAC (with ϵ infinitesimal),

$$\delta L=\epsilon\sum_i m_i x_i=\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i x_i. \tag{36}$$

Thus equation (2) indeed holds, where

$$S=\sum_i m_i x_i. \tag{37}$$

So the boost group is associated with a conservation, equation (7), where the conserved quantity is

$$Q=M(X-\dot{x}t) \tag{38}$$

(up to a constant), where M is the total mass,

$$M=\sum_i m_i, \tag{39}$$

X is the coordinate of the centre of mass,

$$X=\sum_i m_i x_i/M, \tag{40}$$

and $\dot{X}$ is the velocity of the centre of mass. To get this we used equation (8) above and equation (76) of SAC, giving

$$\Pi=t\sum_i m_i x_i. \tag{41}$$

Conservation of Q is equivalent to conservation of centre-of-mass velocity.

So far we have exhibited the operation of Noether's theorem for this example. Concerning the rest of § 2, we notice that equation (9) does not hold, and we look for another member of the Noether family of the boost group for which equation (9) does hold. For this member, $(\bar{M}_{\epsilon,t}, \bar{f}_\epsilon)$, equation (12) (or (15)) must hold in the form

$$f'=MX/L, \tag{42}$$

due to equations (35), (37), (39), (40), and $\bar{M}_{\epsilon,t}$ is obtained from equation (11). However, since we can represent $N(t, t_1)$ explicitly only for infinitesimal time intervals, we can use equation (11) only by taking ϵ infinitesimal, to ensure that $\bar{f}_\epsilon$ and f_ϵ, which then differ infinitesimally from t individually, will differ infinitesimally from each other. Then

$$\bar{f}_\epsilon=t+\epsilon MX/L. \tag{43}$$

As long as ϵ is infinitesimal, we know that $\bar{f}_\epsilon$ is strictly monotonically ascending in t. Equation (11) then becomes

$$\bar{M}_{\epsilon,t}(x_i(t), \dot{x}_j(t)) = N((t+\epsilon MX/L), t)M_{\epsilon,t}(x_i(t), \dot{x}_j(t)), \tag{44}$$

so that

$$(x_i(t), \dot{x}_j(t)) \xrightarrow{M_{\epsilon,t}} (x_i(t) + \epsilon(t + \dot{x}_i(t)MX/L), \dot{x}_j(t) + \epsilon(1 + (F_j/m_j)MX/L)) \tag{45}$$

by equations (31) and (32).

We now turn to § 3 and the inverse Noether's theorem. This time we start with the Lagrangian of equation (28), equations of motion (29) and (30), the temporal development mapping of equation (31), and with the conservation of Q, where Q is given by equation (38). Indeed, we know that a variation of $N(t, t_1)$, $N_\epsilon(t, t_1)$, can be found for which equation (19) (or (24)) holds. It is given by equation (34), and for it

$$L' = t \sum_i E^i L \tag{46}$$

by equations (29), (30), (38)–(40) above and equation (77) of SAC. Thus, we obtain via equation (25) that the boost group of equations (32) and (33) is a symmetry, as are all members of its Noether family, of which it is, in fact, the time-non-varying member. By equation (26) it is also a member for which equation (90) of SAC holds.

5. Summary of CTDNT

Our results concerning the inverse Noether's theorem together with the CTDNT results found in SAC can now be summarised. (The summaries presented in SAC in the next-to-last paragraph of § 4 and in the second paragraph of § 8 do not include the inverse Noether's theorem results that were just derived for the generalisation of CTDNT.) We can state the following.

CTDNT (in its generalised version) allows the association of certain one-parameter continuous groups of symmetry transformations and conservations. Not every symmetry group needs to be associated with a conservation (since equation (42) of SAC might hold rather than equation (41) of SAC or its generalisation, equation (2) above). Not every conservation needs to be associated with a transformation group but, if it is, it is associated with a Noether family of symmetry groups.

6. General symmetry–conservation formalism

The evidence before us consists of LTDCE amd CTDNT, where both approaches were distilled from well-known and useful formalisms for expressing the laws of nature. Both LTDCE and CTDNT associate symmetries and conservations, each in its own way. Indeed, these two ways of associating symmetries and conservations are so different from each other that they seem to have nothing significant in common. Yet they certainly can coexist peacefully, since both LTDCE and CTDNT are parts of the foundation of quantum theory.

However, we want more than that. We would, in fact, like to have a general symmetry–conservation principle, of which both LTDCE and CTDNT, as different as they might be, are realisations. Such a principle, if it existed, would necessarily be very abstract, since it would be an abstraction of both LTDCE and CTDNT, which are already quite abstract themselves. We have not (yet) found a general symmetry principle. For the present we make do with the following proposal of a general symmetry–conservation formalism.

Recall that a symmetry group of state space (not necessarily a symmetry of the laws of nature) defines a decomposition of state space into equivalence subspaces, each equivalence subspace consisting of all states that are related with each other by any element of the symmetry group, i.e. each equivalence subspace is an orbit of the symmetry group in state space. On the other hand, an equivalence relation in state space defines a symmetry group (again not necessarily a symmetry of the laws of nature), which is the maximal subgroup of the transformation group of state space that preserves equivalence subspaces. (If non-invertible transformations are of interest, we may speak of the transformation semigroup of state space, and an equivalence relation will define a symmetry semigroup.)

In this sense a conservation can be considered to be a symmetry (not of the laws of nature!). A conserved quantity decomposes state space into equivalence subspaces, where each equivalence subspace consists of all states having the same value of the conserved quantity. The temporal development mapping, N, preserves these equivalence subspaces and can therefore be considered a symmetry transformation (or family of symmetry transformations) and, if N is invertible, an element (or subset or subgroup) of the symmetry group defined by this equivalence relation. (If N is non-invertible, it will be an element (or subset or subsemigroup) of the respective symmetry semigroup. In our discussion of LTDCE in § 2 of SAC we found that for N non-invertible there arises the possibility of partial conservation.) Any conservation, whether of a 'quantity' or not, can be described in this way. So in our relentless quest for generalisation and abstraction we now introduce the general, abstract definition of conservation (setting aside partial conservation for the time being). Conservation is defined by N preserving a decomposition of state space into equivalence subspaces. We can then speak of conservation of the equivalence relation or of the decomposition.

Let us now consider the general formalism defined by the following three axioms.

(i) There exists a mapping from the set of discrete invertible transformations and one-parameter transformation groups of state space to the set of decompositions of state space into equivalence subspaces (or the set of equivalence relations in state space). (Not every element of the former set needs to be an object of the mapping, and not every decomposition (or equivalence relation) needs to be an image.) In other words, each discrete invertible transformation or one-parameter transformation group of state space might or might not define a decomposition of state space into equivalence subspaces and, if it does, it will do so uniquely. Denote an element of the former set by M, an element of the latter set by Q, and the mapping by $M \to Q$.

(ii) If M is a symmetry of the laws of nature ($MN = NM$) and $M \to Q$, there is conservation of Q (N preserves Q). In other words, if a discrete invertible transformation or a one-parameter transformation group of state space is a symmetry of the laws of nature, then the decomposition of state space into equivalence subspaces defined by it, if it defines such a decomposition, is preserved by N.

(iii) If there is conservation of Q (N preserves Q) and there exists one or more M such that $M \to Q$, then all such M are symmetries of the laws of nature ($MN = NM$). In other words, if a decomposition of state space into equivalence subspaces is preserved by N, then all discrete invertible transformations and one-parameter transformation groups of state space that define this decomposition, if there are any, are symmetries of the laws of nature.

Both LTDCE and CTDNT are compatible with this formalism. For LTDCE (§ 2 of SAC) assumption (a) ('multiplication by a real number') allows fulfilment of axiom (i). If M is a discrete invertible transformation, it will define a decomposition of state space into equivalence subspaces by having all the eigenstates of M belonging to a common eigenvalue form an equivalence subspace for each eigenvalue, and having all the states that are not eigenstates of M form another equivalence subspace. If M is a one-parameter transformation group, it will define a decomposition similarly, but with eigenstates and eigenvalues of its generator, $\mathrm{d}M_\epsilon/\mathrm{d}\epsilon|_{\epsilon=0}$. The mapping $M \to Q$ will be many-to-one, if different M and $\mathrm{d}M_\epsilon/\mathrm{d}\epsilon|_{\epsilon=0}$, although they do not have to have the same eigenvalues, have the same sets of eigenstates belonging to common eigenvalues.

Assumptions (b) (N linear) and (c) (N invertible) (together with (a)) then allow fulfilment of axiom (ii), while assumptions (b) and (d) (eigenstates of M form a complete set) (together with (a)) allow fulfilment of axiom (iii). (If symmetry follows from partial conservation, it will follow from conservation *a fortiori*.), as detailed in SAC. Thus LTDCE with all of assumptions (a), (b), (c) and (d) fits nicely into the general formalism.

For CTDNT discrete transformations are not relevant at all, so they do not serve as objects of the mapping $M \to Q$ of axiom (i). Among the one-parameter transformation groups $(M_{\epsilon,t}, f_\epsilon(u, t))$ only those for which equation (2) holds define decompositions of state space into equivalence subspaces, which is done by means of the function $Q_{(M,f)}(u, t)$ of equation (8). The decomposition is obtained by having each equivalence subspace at time t consist of all states u for which $Q_{(M,f)}(u, t)$ has the same value. That fulfils axiom (i).

As for axiom (ii), for a symmetry group for which equation (2) holds (i.e. equation (42) of SAC does not hold) $Q_{(M,f)}(u, t)$ is indeed conserved (Noether's theorem). And axiom (iii) is also fulfilled, since if $Q_{(M,f)}(u, t)$, related to $(M_{\epsilon,t}, f_\epsilon(u, t))$ by equations (2) and (8), is conserved, the transformation group is a symmetry (inverse Noether's theorem).

However, although CTDNT, as we just saw, fits nicely into the general formalism, it does so very tightly, since in axiom (i) the mapping $M \to Q$ is defined only for symmetry groups associated with conservations and not for a wider class of objects, as in the case of LTDCE. Thus axioms (ii) and (iii) are fulfilled automatically by fulfilling axiom (i). Attempts to make CTDNT fit more loosely into the general formalism seem to run into difficulties.

For example, if instead of the $Q_{(M,f)}(u, t)$ of equation (8) we use

$$Q_{(M,f)}(u, t) = -f'(u, t)L(u, t) - \Pi_{(M,f)}(u, t) \tag{47}$$

to allow a transformation group $(M_{\epsilon,t}, f_\epsilon(u, t))$ to define a decomposition of state space into equivalence subspaces, as above, all transformation groups will be objects of the mapping $M \to Q$ of axiom (i). However, axioms (ii) and (iii) will then be fulfilled only for symmetry groups obeying equation (9), and symmetry groups obeying equation (2) but not equation (9) will not be covered. That could be acceptable, if we knew that every Noether family contains a member obeying equation (9). But, as we saw in § 2, although the possibility exists, it does not seem to be guaranteed.

Or instead of the $Q_{(M,f)}(u, t)$ of equation (8) or (47) we could use

$$\begin{aligned} Q_{(M,f)}(u, t) &= \frac{1}{\epsilon}\int_{t_0}^{t} \mathrm{d}t\, \delta_{(M,f)}L - \Pi_{(M,f)}(u, t) \\ &= \frac{1}{\epsilon}\int_{t_0}^{t} \mathrm{d}t\, [\delta_{(M,f)}L] - f'(u, t)L(u, t) - \Pi_{(M,f)}(u, t), \end{aligned} \tag{48}$$

where t_0 is arbitrary but fixed, for the same purpose. For invertible $N(t, t_1)$ (affording unique integration paths in state space) this gives us a mapping $M \to Q$ for all transformation groups and thus fulfils axiom (i). This function is non-local in general, becoming local only when equation (2) is satisfied. By equations (3) and (6) it is conserved (i.e. equation (7) holds) for all transformation groups, so that axiom (ii) is fulfilled automatically and axiom (iii) not at all. However, if we define what we mean by 'conservation', for the purpose of fulfilling the general formalism, as conservation of a *local* function of state and time, then axioms (ii) and (iii) are fulfilled non-trivially. The difficulty here, if indeed it is a difficulty, is that a non-local function is being used and that this function is defined in general only for invertible $N(t, t_1)$.

7. Conclusions

Our generalisation of CTDNT to more general transformation groups led to the unguaranteed possibility that every Noether family contains a transformation group obeying equation (9), such transformation groups possessing certain advantages.

For the inverse Noether's theorem in generalised CTDNT we found the strong result that every Noether family is a family of symmetry groups of the laws of nature.

We proposed a general symmetry–conservation formalism, into which both LTDCE and CTDNT, as different as they are, fit nicely.

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References

Rosen J 1972 *Ann. Phys., NY* **69** 349
Rosen J and Freundlich Y 1978 *Am. J. Phys.* **46** 1030